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Kelvin wave propagation along an irregular coastline

By L. A. MYSAK AND C. L. TANG
Institute of Oceanography, University of British Columbia, Vancouver

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We discuss the theory of Kelvin wave propagation along an infinitely long coastline which is straight except for small deviations which are treated as a stationary random function of distance along the coast. An operator expansion technique is used to derive the dispersion relation for the coherent Kelvin wave field. For the subinertial case $\sigma = \omega/f < 1$ (ω = wave frequency, f = Coriolis parameter), it is shown that the wave speed is always decreased by the coastal irregularities. Moreover, while the coherent wave amplitude is unaltered, the energy flux along the coast is decreased by the irregularities. For the case $\sigma > 1$, however, we show that in the direction of propagation the wave is attenuated (with the energy being scattered into the random Poincaré and Kelvin wave modes) and that the wave speed is again decreased. Applications of the theory are made to the California coast and North Siberian coast to determine the decrease in phase velocity due to small coastal irregularities. For the California coast the percentage decrease is only about 1 %. For the Siberian coast, however, the percentage decrease is about 25 % for the K_1 tide, and a minimum of 25 % for the M_2 tide. The attenuation of a Kelvin wave, however, appears to be due to very large scale irregularities. An estimate of the actual attenuation rate is not possible, though, because of the relatively short extent of coastal contours available for spectral analysis.

Although attention in this paper has been focused on Kelvin wave propagation, the method developed could readily be used to study the behaviour of other classes of waves trapped against a randomly perturbed boundary.

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1. Introduction

The changes in behaviour of a classical Kelvin wave (Lamb 1932, §208) due to departures from an infinitely long and straight coastline have been the subject of many recent investigations. In most of these studies a classical linear diffraction problem is considered: the solution is sought for the diffracted field due to a Kelvin wave incident upon a sharp right-angle corner (Buchwald 1968), a sharp corner of arbitrary angle (Packham & Williams 1968), and an arbitrary distortion of semi-infinite length of a nearly straight coastline (Miles 1972*a*). In an attempt to investigate Kelvin waves travelling along an infinitely long coast (at $x = 0$ say) with small deviations $x = b(y)$, where y is the distance along the coast, Pinsent (1972) employed an ordinary perturbation scheme to determine the first- and second-order corrections to the wave field due to a zeroth-order 'incident' Kelvin wave. However it is now well known (Frisch 1968, p. 85) that such a perturbation approach gives a uniformly valid (i.e. non-secular) approximation to the total field only if the deviations in the coastline have compact support: that is, the equation of the coastline must be of the form

$$x = \begin{cases} a(y), & |y| \leqslant L, \\ 0, & |y| > L, \end{cases}$$

for some suitable length L. Although Pinsent does not say so explicitly, one of the restrictions on his solution is that either the equation of the coastline $x = b(y)$ must be of the above form or else $b(y)$ must tend to zero sufficiently rapidly as $|y| \to \infty$. Otherwise the Fourier transforms defined just before his equation (5.1) would be unbounded.

With a view towards understanding tidal propagation along an *extensive* irregular coastline, we discuss in this paper the *dispersion relation* for the *mean* or *coherent* Kelvin wave field near an infinitely long coast that is straight except for small deviations which are regarded as a *stationary random function* of position along the coast. Thus, in contrast to the earlier work on this topic, we adopt a *stochastic* model since along many continental boundaries the coast is essentially straight with small deviations which are indeed of a very irregular nature. Furthermore, the technique used involves a second-order expansion *not* of the unknown field, but of a certain differential operator occurring in the mathematical formulation of the problem. This procedure is equivalent to a summation of an infinite subseries of secular terms that results from an ordinary perturbation expansion (Frisch 1968, p. 116). This operator expansion approach was developed by Keller (1967) for the purpose of determining the dispersion relation for a coherent wave propagating in an infinite random medium in the absence of boundaries. In §2 below we show how this technique can be modified to determine the dispersion relation for a propagating wave in a semi-infinite uniform medium that is trapped against a randomly perturbed boundary of infinite extent. Although attention in this paper is focused on Kelvin wave propagation, the method outlined below can readily be used to study the properties of other classes of waves trapped against a randomly perturbed boundary. As examples, we mention here Rayleigh and Love waves in seismology, edge and continental-shelf waves in the ocean, electromagnetic surface waves in conductors, and acoustic surface waves in elastic materials and in the atmosphere.

For the sake of completeness we note here that this paper complements the recent article by Howe & Mysak (1973), who discussed the reflexion and scattering of a *Poincaré* wave by an irregular coastline of infinite extent. Within the framework of a *stochastic* model in which the coastline has deviations represented by a stationary random function, they showed that a Kelvin wave can be generated by a Poincaré wave impinging upon the coast from the open ocean. In this paper we now discuss the behaviour of this Kelvin wave as it propagates along such a coastline.

After deriving a general expression for the dispersion relation of a propagating coherent wave trapped against a randomly perturbed boundary in §2, we formulate the appropriate boundary-value problem for Kelvin waves near an irregular coastline (§ 3). The theory in § 2 is then used to determine the dispersion relation for the mean Kelvin wave field. Then, in terms of the spectrum of the coastal fluctuations, we prove in § 5 some quite general results concerning the reality of the dispersion relation, the magnitude of the phase-speed change and the sign of the attenuation rate. These results concur with our physical intuition concerning scattering and wave propagation phenomena. In § 6 simple formulae are derived for the modified phase speed and the attenuation rate in the limit of short and long correlation scales. These in turn are used in § 7 to obtain an estimate of the actual importance of coastal irregularities in the propagation of Kelvin waves along the California coast and North Siberian coast.

2. Dispersion relation for waves travelling along a randomly perturbed boundary

Consider the semi-infinite region $\mathscr{D}$ in the x, y plane:

$$\mathscr{D} = \{x, y \mid x > \epsilon s(y), |y| < \infty\},$$

where $s(y)$ is a stationary random function with zero mean and ϵ is a small positive parameter. Let $\psi(x,y)e^{i\omega t}$ be the wave field of a stochastic system characterized by the elliptic equation

$$L\psi(x,y) = 0 \quad \text{in} \quad \mathscr{D}, \tag{2.1}$$

where L is a linear deterministic differential operator which maps a function defined on $\mathscr{D}$ onto another function defined on $\mathscr{D}$. Further, suppose that for sufficiently small ϵ the boundary conditions can be written in the form

$$\mathscr{B}\psi \equiv (B+C)\psi = F(y) \quad \text{on} \quad \partial\mathscr{D}, \tag{2.2}$$

$$|\psi| \to 0 \quad \text{as} \quad x \to \infty, \tag{2.3}$$

where $\partial\mathscr{D} = \{x, y \mid x = 0, |y| < \infty\}$, and B and $C = C[s]$ are respectively deterministic and stochastic linear differential operators which map functions defined on $\mathscr{D}$ onto functions defined on $\partial\mathscr{D}$, and $F(y)$ is a given deterministic function. For any *realization* of the system characterized by the random function

$$s(y) \equiv s(y;\theta),$$

where $\theta \in \Theta \equiv$ space of events, $\psi \equiv \psi(x,y;\theta)$ is also a random function. It is our purpose to derive the dispersion relation for trapped wave solutions for the mean or coherent field $\langle\psi\rangle$ that is implied by (2.1)–(2.3) when $F = 0$; here angular brackets denote the *ensemble* average over many realizations of the system. For convenience, henceforth we shall always suppress the θ dependence of the random functions.

Since L is deterministic, the average of (2.1) gives

$$L\langle\psi\rangle = 0 \quad \text{in} \quad \mathscr{D}. \tag{2.4}$$

Assuming $\mathscr{B}$ can be inverted, (2.2) yields

$$\langle\psi\rangle = \langle\mathscr{B}^{-1}\rangle F,$$

which in turn implies that

$$\langle \mathscr{B}^{-1}\rangle^{-1}\langle\psi\rangle = F(y) \quad \text{on} \quad \partial\mathscr{D}. \tag{2.5}$$

Averaging (2.3), we find that

$$\langle|\psi|\rangle \to 0 \quad \text{as} \quad x\to\infty. \tag{2.6}$$

Now if L and B are translationally invariant in y, C is statistically homogeneous and $F = 0$, then (2.4)–(2.6) admit *trapped travelling wave solutions*

$$\langle\psi\rangle = e^{-lx+imy}, \quad \operatorname{Re} l > 0,$$

which obey the dispersion relation† given by the *two* equations

$$e^{-q}L^q e = 0 \quad \text{in} \quad \mathscr{D} \tag{2.7}$$

and

$$e^{-imy}\langle\mathscr{B}^{-1}\rangle^{-1}e^q = 0 \quad \text{on} \quad \partial\mathscr{D}, \tag{2.8}$$

where $q = -lx+imy$. Following Keller (1967), we write $\mathscr{B}^{-1}$ as $(I+B^{-1}C)^{-1}B^{-1}$, use the binomial expansion and then average; this gives

$$\langle\mathscr{B}^{-1}\rangle = \{I-B^{-1}\langle C\rangle + B^{-1}\langle CB^{-1}C\rangle + O[(B^{-1}C)^3]\}B^{-1}, \tag{2.9}$$

which is valid provided that the operator norm of $B^{-1}C$ is less than one. Finally, taking the inverse of (2.9) and dropping third- and higher-order terms in $B^{-1}C$, we obtain the dispersion relation

$$e^{-q}Le^q = 0 \quad \text{in} \quad \mathscr{D}, \tag{2.10a}$$

$$e^{-imy}[B+\langle C\rangle + \langle C\rangle B^{-1}\langle C\rangle - \langle CB^{-1}C\rangle]e^q = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{2.10b}$$

Equation (2.10b) can also be derived without introducing the function $F(y)$ by converting the boundary condition into an integral equation. We write the boundary condition (2.2), with $F = 0$, in the form

$$B\psi = -C\psi \quad \text{on} \quad \partial\mathscr{D}.$$

Let ψ_0 be the solution in the absence of coastal irregularities which satisfies

$$L\psi_0 = 0 \quad \text{in} \quad \mathscr{D},$$

$$B\psi_0 = 0 \quad \text{in} \quad \partial\mathscr{D}.$$

Hence the boundary condition for ψ becomes the integral equation

$$\psi = \psi_0 - B^{-1}C\psi.$$

Hence, we obtain

$$\psi = (I+B^{-1}C)^{-1}\psi_0, \tag{2.10c}$$

where ψ is the particular solution satisfying the radiation condition and approaches ψ_0 as $\epsilon\to 0$. On averaging (2.10c), inverting, and operating with B, we obtain

$$B\langle(I+B^{-1}C)^{-1}\rangle^{-1}\langle\psi\rangle = B\psi_0 = 0.$$

Upon using the binomial expansion and introducing the exponentials this clearly reduces to (2.10b).

† By this we mean the two relations $l = l(\omega)$ and $m = m(\omega)$ implied by (2.7) and (2.8).

It is important to note here that B^{-1} is an *integral* operator which maps a function defined on $\partial\mathscr{D}$ onto a function defined on $\mathscr{D}$:

$$B^{-1}f(y) \equiv B^{-1}_{x,y}f(y) = \int_{-\infty}^{\infty} G(x,y;y')f(y')\,dy', \tag{2.11}$$

where
$$\left.\begin{aligned} LG(x,y;y') &= 0 \quad \text{in} \quad \mathscr{D}, \\ BG(x,y;y') &= \delta(y-y') \quad \text{on} \quad \partial\mathscr{D}. \end{aligned}\right\} \tag{2.12}$$

And since C maps functions on $\mathscr{D}$ onto functions on $\partial\mathscr{D}$, it follows that in (2.10*b*) the term $\langle CB^{-1}C\rangle e^{q(x,y)}$ on $\partial\mathscr{D}$, for example, implies the following sequence of operations:

$$\begin{aligned} [\langle CB^{-1}C\rangle e^{-lx+imy}]_{x=0} &= \langle\{C_{x,y}[B^{-1}_{x,y}(C_{x,y}e^{-lx'+imy})_{x'=0}]\}_{x=0}\rangle \\ &= e^{imy}\langle\mathscr{F}[s]\rangle, \end{aligned}$$

where $\mathscr{F}[s]$ is an integro-differential functional of s that is independent of x and y and $C_{x,y}$ denotes an operator with x and y derivatives.

The approximation (2.10*b*) has been derived by a number of authors in connexion with wave propagation in an *infinite* random medium; in such a case this equation holds in $\mathscr{R}^2$ and e^q is replaced by $e^{i\mathbf{k}\cdot\mathbf{x}}$ and e^{-imy} by $e^{-i\mathbf{k}\cdot\mathbf{x}}$ (see, for example, the 'first-order smoothing approximation' of Keller 1967, Frisch 1968, and the 'binary collision approximation' of Howe 1971). Howe, in fact, does consider the case of random boundary-value problems by using both the mean and fluctuating field $\psi' = \psi - \langle\psi\rangle$. After a few simplifications of his equations (6.5)–(6.9), one can deduce our equations (2.10*a*, *b*).

3. Boundary-value problem for Kelvin waves on an irregular coastline

Let (x,y,z) be a right-handed set of rectangular co-ordinates. Then the linearized shallow-water equations for a homogeneous layer of fluid rotating uniformly about the z axis with angular velocity $\frac{1}{2}f$ are given by

$$\left.\begin{aligned} i\omega u - fv + g\zeta_x &= 0, \\ i\omega v + fu + g\zeta_y &= 0, \end{aligned}\right\} \tag{3.1}$$

$$u_x + v_y + i\omega\zeta/h = 0, \tag{3.2}$$

where (u,v) is the horizontal fluid velocity, ζ is the surface elevation of the fluid above the mean depth h, g is the acceleration due to gravity and a time factor $e^{i\omega t}$ has been assumed for each of the dynamic variables. Without loss of generality, we take $f > 0$.

From (3.1) we readily deduce that

$$\left.\begin{aligned} au &= i\omega\zeta_x + f\zeta_y, \\ av &= -f\zeta_x + i\omega\zeta_y, \end{aligned}\right\} \tag{3.3}$$

where $a = (\omega^2 - f^2)/g$; substituting (3.3) into (3.2) we obtain

$$\nabla^2\zeta + (a/h)\,\zeta = 0 \tag{3.4}$$

as the reduced wave equation for the surface elevation.

Suppose that the fluid occupies a semi-infinite region with $x > 0$ of the x, y plane and that it is bounded by an irregular coastline specified by the equation

$$x = b(y), \tag{3.5}$$

where $b(y)$ is a stationary random function with zero mean. We require that the velocity normal to the coast be zero; this implies that

$$u = vb_y \quad \text{on} \quad x = b(y),$$

or, upon using (3.3),

$$i\sigma\zeta_x + \zeta_y - b_y(-\zeta_x + i\sigma\zeta_y) = 0 \quad \text{on} \quad x = b(y), \tag{3.6}$$

where $\sigma = \omega/f$.

If we introduce the horizontal length scale $d = (gh)^{\frac{1}{2}}/f$ and the non-dimensional variables $(x', y', b') = d^{-1}(x, y, b)$, equations (3.4) and (3.6) give, after dropping the primes,

$$L\zeta \equiv (\partial_x^2 + \partial_y^2 + \kappa^2)\zeta = 0, \tag{3.7}$$

$$(B - b_y D)\zeta = 0 \quad \text{on} \quad x = b(y), \tag{3.8}$$

where $\kappa^2 = \sigma^2 - 1, \quad B = \partial_x - (i/\sigma)\partial_y, \quad D = (i/\sigma)\partial_x + \partial_y.$

We assume that the coastal irregularities are small in comparison with d, so that (the non-dimensional) $b(y)$ can be written as

$$b(y) = \epsilon s(y), \tag{3.9}$$

where $s(y)$ is an $O(1)$ stationary random function with zero mean and $0 < \epsilon \ll 1$. Finally, we expand (3.8) in powers of b and obtain the linearized boundary condition

$$(B + C)\zeta = 0 \quad \text{on} \quad x = 0, \tag{3.10}$$

where

$$C = \epsilon(sB\partial_x - s_y D) + \epsilon^2(\tfrac{1}{2}s^2 B\partial_x^2 - s_y sD\partial_x) + O(\epsilon^3). \tag{3.11}$$

Equations (3.7) and (3.10) correspond to (2.11) and (2.2) (with $F = 0$). Hence we are now in a position to determine the dispersion relation for a coherent Kelvin wave of the form

$$\langle\zeta\rangle = e^{-lx + imy}, \tag{3.12}$$

where $\operatorname{Re} l > 0$, in accordance with (2.3). If $\epsilon = 0$, corresponding to a straight coast, (3.7), (3.10) and (3.12) imply that $l = 1$ and $m = \sigma$, the relations for a classical Kelvin wave travelling in the negative y direction with a non-dimensional phase speed of unity.

4. Dispersion relation for a Kelvin wave

Before simplifying (2.10*a*, *b*) for the Kelvin wave case, we derive the appropriate Green's function as defined by (2.12), with L and B given by (3.7) and (3.8) respectively. Let us introduce the Fourier transform of G:

$$\hat{G}(x, \eta; y') = \int_{-\infty}^{\infty} G(x, y; y')\, e^{-i\eta y}\, dy.$$

Then it is easy to show (Buchwald 1971) that, for $\sigma > 1$,

$$G(x, y; y') = G(x, y - y') = \frac{1}{2\pi}\int_\Gamma d\eta\, \frac{\sigma}{\eta - \sigma(\eta^2 - \kappa^2)^{\frac{1}{2}}} \exp[-(\eta^2 - \kappa^2)^{\frac{1}{2}} x + i\eta(y - y')], \tag{4.1}$$

where Γ is a contour along the real η axis indented below the branch point $\eta = -\kappa$ and above the branch point $\eta = \kappa$ and simple pole $\eta = \sigma$. With these indentations and the branch cut from $\eta = -\kappa\,(+\kappa)$ extending to infinity in the upper (lower) half η plane, the radiation condition is satisfied. For the case $\sigma < 1$, the solution can be obtained from (4.1) by the transformations $\kappa \to -i\kappa'$, where $\kappa' = (1-\sigma^2)^{\frac{1}{2}} > 0$, and $\Gamma \to \Gamma'$, a contour along the real axis indented above the pole $\eta = \sigma$.

For a Kelvin wave, (2.10*a*) yields

$$l^2 - m^2 + \sigma^2 - 1 = 0. \tag{4.2}$$

From (3.11) we deduce that

$$\langle C \rangle = \tfrac{1}{2}\epsilon^2 R(0)\, B\,\partial_x^2 + O(\epsilon^3),$$

where $R(y)$ is the *covariance function* defined by

$$R(y) = \langle s(y+y')\, s(y') \rangle. \tag{4.3}$$

Hence, correct to $O(\epsilon^2)$, equation (2.10*b*) implies that

$$B_1 + B_2 + B_3 = 0, \tag{4.4}$$

where $$B_1 = -l + m/\sigma, \quad B_2 = \tfrac{1}{2}\epsilon^2 R(0)\,(-l+m/\sigma)\, l^2, \qquad (4.5), (4.6)$$

$$B_3 = -\epsilon^2 e^{-imy} \langle (sB\,\partial_x - s_y D)\, B^{-1} (sB\,\partial_x - s_y D) \rangle\, e^q |_{x=0}. \tag{4.7}$$

After some lengthy algebra, (4.7) reduces to

$$B_3 = -\epsilon^2 \int_{-\infty}^{\infty} dy\, e^{-imy} \{ [G_{xx}(0,y) + (m/\sigma)\, G_x(0,y)]\, P(y) + G(0,y)\,[-imP'(y) + P''(y)] \}, \tag{4.8}$$

where $$P(y) = -\alpha l R(y) + i\beta R'(y), \quad \alpha = -l + m/\sigma, \quad \beta = -l/\sigma + m. \tag{4.9}$$

Equations (4.2) and (4.4) represent two coupled equations for l and m, from which we seek functions of the form $l = l(\sigma)$ and $m = m(\sigma)$. Since B_2 and B_3 are $O(\epsilon^2)$, it is convenient to put

$$l = l_0 + \epsilon^2 l_1, \quad m = m_0 + \epsilon^2 m_1, \tag{4.10}$$

where $(l_0, m_0) = (1, \sigma)$ are the deterministic Kelvin wave relations given at the end of §3. On substituting (4.10) into (4.2), we get, correct to $O(\epsilon^2)$,

$$l_1 = \sigma m_1. \tag{4.11}$$

On substituting (4.10) into (4.5), (4.6) and (4.8) and noting that

$$\alpha(l_0, m_0) = 0 \quad \text{and} \quad \beta(l_0, m_0) = (\sigma^2 - 1)/\sigma$$

[see (4.9)], we find that (4.4) reduces to

$$-l_1 + m_1/\sigma - [(\sigma^2-1)/\sigma] \int_{-\infty}^{\infty} dy\, e^{-i\sigma y} \{ G(0,y)\,[\sigma R''(y) + iR'''(y)] + [G_x(0,y) + G_{xx}(0,y)]\, iR'(y) \} = 0, \tag{4.12}$$

which is correct to $O(\epsilon^2)$. Finally, if we use (4.11) to eliminate l_1 in (4.12), we get, for $\sigma \neq 1$,

$$m_1 = -\int_{-\infty}^{\infty} dy\, e^{-i\sigma y} \{\text{as in (4.12)}\}. \tag{4.13}$$

Since $R(y)$ and $G(0,y)$ are in principle known functions, the first non-trivial corrections to l_0 and m_0 due to the coastal irregularities are thus determined by (4.11) and (4.13). But since only a Fourier integral representation of $G(0,y)$ is available [see (4.1)], (4.13) is not in a convenient form for further analysis. However, if we invoke the Wiener–Khinchin theorem, which states that for *ergodic* stationary random processes the energy spectral density $S(\eta)$ or 'spectrum' is simply the Fourier transform of the covariance function $R(y)$, as defined by (4.3), viz.

$$S(\eta) = \mathscr{F}\{R(y)\} = \hat{R}(\eta) = \int_{-\infty}^{\infty} e^{-i\eta y} R(y)\, dy, \tag{4.14}$$

then (4.13) can be considerably simplified by using the *convolution theorem* for Fourier integrals:

$$\begin{aligned}\mathscr{F}\{h(y)\,k(y)\} &= \int_{-\infty}^{\infty} e^{-i\sigma y}\, h(y)\, k(y)\, dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\sigma-\xi)\,\hat{k}(\xi)\, d\xi,\end{aligned} \tag{4.15}$$

where $\hat{h}(\xi) = \mathscr{F}\{h\}$ and $\hat{k}(\xi) = \mathscr{F}\{k\}$. From (4.1) it follows that

$$\left.\begin{aligned}\mathscr{F}\{G(0,y)\} &= \hat{G}(0,\eta) = \sigma/[\eta-\sigma(\eta^2-\kappa^2)^{\frac{1}{2}}], \\ \mathscr{F}\{G_x(0,y)\} &= -(\eta^2-\kappa^2)^{\frac{1}{2}}\hat{G}(0,\eta), \\ \mathscr{F}\{G_{xx}(0,y)\} &= (\eta^2-\kappa^2)\,\hat{G}(0,\eta)\end{aligned}\right\} \tag{4.16}$$

for $\sigma > 1$; for $\sigma < 1$, $\kappa \to -i\kappa'$ in (4.16). Also (4.14) implies that

$$\mathscr{F}\{R^{(n)}(y)\} = (i\eta)^n\,\hat{R}(\eta) = (i\eta)^n S(\eta). \tag{4.17}$$

Hence on applying (4.15) to (4.13) and using (4.16) and (4.17), we obtain, for $\sigma > 1$,

$$\begin{aligned} m_1 &= -\frac{1}{2\pi}\int_C d\xi \frac{\sigma S(\xi)}{\sigma-\xi-\sigma[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}}}\{\sigma(i\xi)^2 + i(i\xi)^3 \\ &\qquad + [-[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}} + (\sigma-\xi)^2-\kappa^2]\, ii\xi\} \\ &= \frac{1}{2\pi}\int_C d\xi\sigma\xi S(\xi)\, A(\xi), \end{aligned} \tag{4.18}$$

where

$$A(\xi) = \frac{[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}} + \sigma\xi - 1}{\sigma[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}} - (\sigma-\xi)}.$$

Since the η and ξ planes are related by the equation $\xi = \sigma-\eta$, the path C in (4.18) is indented *below* the branch point at $\xi = \sigma-\kappa$ ($\eta = \kappa$) and *above* the branch point at $\xi = \sigma+\kappa$ ($\eta = -\kappa$). Since the integrand has a simple zero at $\xi = 0$ ($\eta = \sigma$), C is not indented at $\xi = 0$. For $\sigma < 1$, it follows that

$$m_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\sigma\xi S(\xi)\frac{[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}} + \sigma\xi - 1}{\sigma[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}} - (\sigma-\xi)}. \tag{4.19}$$

5. Consequences of Kelvin wave dispersion relation

We now deduce some very general results which follow from (4.18) and (4.19).

(1) *For $\sigma < 1$, m_1 and hence l_1 [see (4.11)] are real.* Since $[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}} > 0$ for $\sigma < 1$ and $S(\xi)$ is real and positive, this result follows immediately from (4.19).

This result is not unexpected on physical grounds in the sense that, when $\sigma < 1$, $\kappa^2 < 0$ and hence (3.7) does not have unattenuated plane wave (i.e. Poincaré wave) solutions. Therefore any energy that is initially trapped against the coast in the form of a Kelvin wave cannot be radiated away from the coast. However, it is important to note that, because of the modification of the dispersion relation, the group velocity and hence the energy flux of the coherent wave differ from the corresponding quantities for a wave travelling along a smooth coast. In fact, we show below (result 3) that the presence of the coastal irregularities decreases this energy flux. Since backscatter up the coast and radiation away from the coast are not possible, this decrease in energy flux is balanced by a corresponding increase in the energy flux associated with the fluctuating Kelvin wave field. (For further elaboration on this for conservative systems, see Howe 1973.)

(2) *For $\sigma < 1$, $m_1 > 0$ and hence the phase speed $c < c_0 = 1$ (the deterministic value)* and $l > l_0 = 1$. To establish that $m_1 > 0$, we first rationalize the denominator in the integrand of (4.19) by multiplying the numerator and denominator by $\sigma[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}+\sigma-\xi$. This gives

$$m_1 = \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} d\xi\, \xi S(\xi)\, H(\xi), \tag{5.1}$$

where

$$H(\xi) = \{-1+[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}\}/(\xi-2\sigma). \tag{5.2}$$

For real ξ it is easy to show that $H(\xi)$ is a monotonically increasing function which is negative for $\xi < 0$ and positive for $\xi > 0$. In fact $H \to -1$ as $\xi \to -\infty$, $H \to 1$ as $\xi \to \infty$ and $H(0) = 0$. Hence the integrand of (5.1) is positive for all ξ since $\xi S(\xi) \gtrless 0$ for $\xi \gtrless 0$. Hence $m_1 > 0$.

Since $m_0 = \sigma$, the phase speed $c = \sigma/(\sigma+\epsilon^2 m_1) = 1/(1+\epsilon^2 m_1/\sigma) < 1$ for $m_1 > 0$. Similarly, the attenuation scale normal to the coast is increased since

$$l = l_0+\epsilon^2 l_1 = 1+\epsilon^2\sigma m_1 > 1.$$

The slowing down of the mean Kelvin wave by the coastal irregularities is also in agreement with physical intuition since in travelling between two points on the coast, the effective coastline is now much longer than in the deterministic case and hence the travel time of the wave is effectively increased.

(3) *For $\sigma < 1$, the group velocity $c_g < 1$ and the energy flux $F < F_0$, where F_0 is the energy flux in the absence of coastal irregularities.* The group velocity (in the negative y direction) c_g is evaluated from the relation $c_g = d\sigma/dm$. Since $m(\sigma) = \sigma+\epsilon^2 m_1(\sigma)$, differentiating both sides of this equation gives

$$c_g = 1/(1+\epsilon^2 dm_1/d\sigma).$$

Using (5.1) and (5.2), we have

$$\frac{dm_1}{d\sigma} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\,\xi S(\xi)\,P(\xi),$$

where $$P(\xi) = \frac{\xi[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}\{[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}-1\}+\sigma\xi(2\sigma-\xi)}{(2\sigma-\xi)^2[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}}.$$

It can be shown that $P(\xi)$ has a similar signature behaviour to $H(\xi)$ as a function of ξ. Thus $dm_1/d\sigma > 0$, which implies that $c_g < 1$. The energy flux (in the negative y direction) of the coherent Kelvin wave is given by

$$F = c_g E, \tag{5.3}$$

where E is the energy density per unit length of coast. In terms of the energy density $\hat{E}$ per unit area, we have

$$E = \frac{(gh)^{\frac{1}{2}}}{f}\int_0^{\infty} \hat{E}\,dx, \tag{5.4}$$

where $$\hat{E} = \tfrac{1}{2}\rho h\{|\langle u\rangle|^2+|\langle v\rangle|^2\}+\tfrac{1}{2}\rho g|\langle\xi\rangle|^2. \tag{5.5}$$

The average values of u and v can be calculated from (3.3). Using $\langle\xi\rangle = e^{-lx+imy}$ and after some algebra, we have, to $O(\epsilon^2)$,

$$\langle u\rangle = -\frac{igm_1\epsilon^2}{(gh)^{\frac{1}{2}}}e^{-lx+imy}, \quad \langle v\rangle = -\frac{g}{(gh)^{\frac{1}{2}}}e^{-lx+imy}. \tag{5.6}$$

Combining (5.3)–(5.6) gives

$$F = (\rho g^2 h/2f)\,[1-(dm_1/d\sigma+\sigma m_1)\,\epsilon^2+O(\epsilon^4)].$$

Since both $dm_1/d\sigma$ and m_1 are positive, we conclude that

$$F < F_0 = \rho g h^2/2f.$$

This result shows that a Kelvin wave will lose energy through scattering when it encounters an extensive irregular coastline.

(4) *For $\sigma > 1$, m_1 is complex and hence $l_1 = \sigma m_1$ is complex, with*

$$\mu_1 \equiv \operatorname{Re} m_1 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\sigma-\kappa}+\int_{\sigma+\kappa}^{\infty}\right\}d\xi\,\sigma\xi S(\xi)\,A(\xi)+\frac{1}{2\pi}\int_{\sigma-\kappa}^{\sigma+\kappa}d\xi\frac{\sigma\xi S(\xi)}{2\sigma-\xi}, \tag{5.7}$$

$$\mu_2 = \operatorname{Im} m_1 = -\frac{1}{2\pi}\int_{\sigma-\kappa}^{\sigma+\kappa}d\xi\frac{\sigma\xi S(\xi)\,[\kappa^2-(\sigma-\xi)^2]^{\frac{1}{2}}}{2\sigma-\xi}. \tag{5.8}$$

For $\xi > \sigma+\kappa$ and $\xi < \sigma-\kappa$, the integrand in (4.18) is real and hence the first term in (5.7) is real. For $\sigma-\kappa < \xi < \sigma+\kappa$, $[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}} = i[\kappa^2-(\sigma-\xi)^2]^{\frac{1}{2}}$ for the branch-cut configuration discussed in the sentence following (4.18). Upon rationalizing $A(\xi)$ for this range of ξ and separating into real and imaginary parts, we readily obtain the second term in (5.7) and the expression for μ_2 in (5.8).

As a consequence of result 4, the Kelvin wave crests are no longer perpendicular to the mean shoreline $x = 0$, and there now exists the possibility of attenuation or growth in the y direction. Putting $m = \sigma+\epsilon^2(\mu_1+i\mu_2)$ and $l = 1+\epsilon^2\sigma(\mu_1+i\mu_2)$

into (3.12), and incorporating the time dependence in the form $e^{i\sigma t}$, where t is the non-dimensional time, we obtain

$$\langle\zeta\rangle = \exp\{-(1+\epsilon^2\sigma\mu_1)x-\mu_2 y-i[\epsilon^2\sigma\mu_2 x-(\sigma+\epsilon^2\mu_1)y-\sigma t]\}. \tag{5.9}$$

Since $0 < \kappa < \sigma$ and hence $0 < \sigma-\kappa < \xi < \sigma+\kappa < 2\sigma$, it follows that the integrand in (5.8) is positive. Hence we have the following result.

(5) *For* $\sigma > 1$, $\mu_2 \equiv \mathrm{Im}\, m_1 < 0$.

As a consequence of result 5 and (5.9) we note that the wavenumber vector $(\epsilon^2\sigma\mu_2, -\sigma-\epsilon^2\mu_1)$ is directed towards the third quadrant in wavenumber space. Thus, as the mean Kelvin wave propagates along the coast in the negative y direction, the wave crests are turned slightly *towards* the coast. Further, we observe that the mean wave is continually attenuated as it propagates along the coast. This behaviour is again understandable on physical grounds upon recalling that, for the case $\sigma > 1$, Poincaré waves can now exist and can radiate energy off to infinity *away* from the coast. Since the *mean* Kelvin wave is turned towards the coast, part of its energy is continually being scattered into the fluctuating Poincaré field (and fluctuating Kelvin field) and consequently the mean amplitude must decrease in the direction of propagation. These remarks are again in keeping with the general result of Howe (1973), who showed that, for conservative systems in an infinite random medium, there is always a net transfer of energy from the mean field to the fluctuating field.

(6) *For* $\sigma > 1$, $\mu_1 \equiv \mathrm{Re}\, m_1 > 0$ *and hence the phase speed* $c < 1$. Since the second term in (5.7) is obviously positive, we simply have to show that the first term is positive. If $A(\xi)$ in (4.18) is rationalized, we find that

$$A(\xi) = \{1-[(\sigma-\xi)^2-\kappa^2]^{\frac{1}{2}}\}/(2\sigma-\xi). \tag{5.10}$$

It is fairly easy to show from (5.10) that for $-\infty < \xi < \sigma-\kappa$

$$A(\xi)\begin{cases} < 0, & -\infty < \xi < 0, \\ = 0, & \xi = 0, \\ > 0, & 0 < \xi < \sigma-\kappa. \end{cases}$$

In fact $A(\xi)$ is a monotonically increasing function with $A(-\infty) = -1$ and $A(\sigma-\kappa) = 1/(\sigma+\kappa)$ for this range of ξ. For $\sigma+\kappa < \xi < \infty$, $A(\xi)$ is a monotonically decreasing function that is always positive with $A(\sigma+\kappa) = 1/(\sigma-\kappa)$ and $A(\infty) = 1$. Hence the integrand in the first term in (5.7) is positive for

$$-\infty < \xi < \sigma-\kappa \quad \text{and} \quad \sigma+\kappa < \xi < \infty$$

and hence $\mu_1 > 0$.

From (5.9) we see that the phase speed is given by

$$\begin{aligned} c &= \sigma/[\epsilon^4\sigma^2\mu_2^2+(\sigma+\epsilon^2\mu_1)^2]^{\frac{1}{2}} \\ &= 1/[\epsilon^4\mu_2^2+(1+\epsilon^2\mu_1/\sigma)^2]^{\frac{1}{2}} \\ &< 1 \quad \text{since} \quad \mu_1 > 0. \end{aligned}$$

6. Asymptotic solutions

The expressions for m_1, namely (4.18) and (4.19), can be simplified considerably if the correlation scale $\mathscr{L}$ measured in units of the length scale $(gh)^{\frac{1}{2}}/f$ is either very short or very long. We calculate separately the asymptotic forms of m_1 for the following four cases: (i) $\sigma < 1, \mathscr{L} \gg 1$; (ii) $\sigma < 1, \mathscr{L} \ll 1$; (iii) $\sigma > 1, \mathscr{L} \gg 1$; (iv) $\sigma > 1, \mathscr{L} \ll 1$.

(i) $\sigma < 1, \mathscr{L} \gg 1$. From (4.19), we have

$$m_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\,\sigma\xi S(\xi)\,A^*(\xi), \tag{6.1}$$

where
$$A^*(\xi) = \frac{[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}+\sigma\xi-1}{\sigma[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}-\sigma+\xi},$$

which can be simplified to the following form after rationalization:

$$A^*(\xi) = \{1-[(\sigma-\xi)^2+\kappa'^2]^{\frac{1}{2}}\}/(2\sigma-\xi). \tag{6.2}$$

Since $S(\xi)$ is the Fourier transform of $R(y)$, a long correlation scale corresponds to a short range of ξ in $S(\xi)$. In other words, if $R(y) \simeq 0$ for $|y| > \mathscr{L} \gg 1$, then $S(\xi) \simeq 0$ for $|\xi| > k$, where the cut-off wavenumber k is of order $1/\mathscr{L} \ll 1$. Consequently, we can estimate the integral by expanding A^* in a power series about $\xi = 0$ and calculate the contribution from the lowest-order term in the series. Thus we have

$$A^*(\xi) = \tfrac{1}{2}\xi + O(\xi^2)$$

and hence
$$m_1 \simeq \frac{\sigma}{4\pi}\int_{-\infty}^{\infty} d\xi\,\xi^2 S(\xi). \tag{6.3}$$

(ii) $\sigma < 1, \mathscr{L} \ll 1$. Contrary to the previous case, $k \gg 1$ for a short correlation scale. The integral in (6.1) can be separated into three parts:

$$m_1 = \frac{\sigma}{2\pi}\left\{\int_{-1}^{1} + \int_{-\infty}^{-1} + \int_{1}^{\infty}\right\} d\xi\,\xi S(\xi)\,A^*(\xi).$$

For the first integral, we expand ξA^* about $\xi = 0$, and for the second and the third ones, we expand about $r \equiv 1/\xi = 0$. We thus find

$$m_1 = \frac{\sigma}{4\pi}\int_{-1}^{1} d\xi\,\xi^2 S(\xi)\,[1+O(\xi)] + \frac{\sigma}{2\pi}\left\{\int_{0}^{-1} + \int_{1}^{0}\right\}\left(-\frac{dr}{r^2}\right) S\left(\frac{1}{r}\right)\left[\frac{1}{|r|} - 1 + \frac{\sigma r}{|r|} + O(r)\right],$$

where it is assumed that $S \propto r^{2+\alpha}$ $(\alpha > 0)$ as $r \to 0$ so that the second and third integrals converge. Thus we have

$$\begin{aligned} m_1 &\simeq \frac{\sigma}{4\pi}\int_{-1}^{1} \xi^2 S(\xi)\,d\xi + \frac{\sigma}{2\pi}\left\{\int_{-\infty}^{-1} + \int_{1}^{\infty}\right\} d\xi\,|\xi| S(\xi) \\ &\simeq \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} |\xi| S(\xi)\,d\xi = \frac{\sigma}{\pi}\int_{0}^{\infty} \xi S(\xi)\,d\xi. \end{aligned} \tag{6.4}$$

In arriving at the final form in (6.4) we have neglected the integral from -1 to 1 and replaced $\int_{-\infty}^{-1} + \int_{1}^{\infty}$ by $\int_{-\infty}^{\infty}$. This is justifiable owing to the long-range nature of $S(\xi)$. That is, since $k \gg 1$ for this case, $S(\xi) = O(1)$ over the extensive intervals $-k < \xi < -1$ and $1 < \xi < k$. We note that, in both the above cases, the expressions (6.3) and (6.4) are real and positive, as required by results 1 and 2.

(iii) $\sigma > 1$, $\mathscr{L} \gg 1$. For $\sigma > 1$, m_1 is given by (6.1) with $A^*(\xi)$ replaced by $A(\xi)$ [see (4.18)], i.e. κ'^2 replaced by $-\kappa^2$. The two branch points of $A(\xi)$ in the ξ plane are at $\sigma - \kappa\ (>0)$ and $\sigma + \kappa\ (>0)$. Thus, for $\xi < \sigma - \kappa$ and $\xi > \sigma + \kappa$, $A(\xi) = A^*(\xi)$. For a sufficiently narrow-ranged $S(\xi)$, the major contribution to the integral comes from ξ around zero, where $A(\xi) = A^*(\xi)$. Hence m_1 has the same asymptotic expression as that in case (i), namely, (6.3). The fact that m_1 is real in this limiting case is not surprising on physical grounds since a large $\mathscr{L}$ would correspond to a relatively *smooth* coastline, in which case scattering and attenuation effects are expected to be negligible.

(iv) $\sigma > 1$, $\mathscr{L} \ll 1$. The general expression for the real part of m_1 is given by (5.7). In the large-k limit, the range of integration of the third term is very small compared with those of the first two terms. We can thus neglect the third term in this calculation.

As noted above, $A(\xi) = A^*(\xi)$ outside $\sigma - \kappa < \xi < \sigma + \kappa$. We can define

$$\bar{A}(\xi) = \begin{cases} A^*(\xi), & \xi < \sigma - \kappa, \quad \xi > \sigma + \kappa, \\ 0, & \sigma - \kappa < \xi < \sigma + \kappa, \end{cases}$$

and write (5.3) as

$$\mu_1 \simeq \frac{\sigma}{2\pi} \int_{-\infty}^{\infty} d\xi\, \xi S(\xi)\, \bar{A}(\xi). \tag{6.5}$$

We then follow the same procedure as in case (ii) to evaluate the integral. The result turns out to be the same as that of (ii). This is due to the fact that the dominant contribution to the integral comes from $|\xi| \gg 1$ in $\bar{A}(\xi)$, where $\bar{A}$ is equal to A^*.

The imaginary part of m_1 is given by (5.8), i.e.

$$\mu_2 = -\frac{\sigma}{2\pi} \int_{\sigma-\kappa}^{\sigma+\kappa} d\xi\, \xi S(\xi) \frac{[\kappa^2 - (\sigma - \xi)^2]^{\frac{1}{2}}}{2\sigma - \xi}.$$

$S(\xi)$ can be treated as constant over the small interval of integration 2κ, and hence can be approximated by $S(\sigma)$. A simple calculation then gives

$$\mu_2 \simeq -\tfrac{1}{4}\sigma(\sigma - 1)(3\sigma - 1)\, S(\sigma).$$

These results are summarized in table 1 below.

7. Applications to California coast and Siberian coast

From the results of the last section, we see that the magnitudes of the irregularities and the correlation scale of a coastline relative to the horizontal length scale $(gh)^{\frac{1}{2}}/f$ are the main factors in determining the corrections to the phase

	$\sigma < 1$	$\sigma > 1$
$\mathscr{L} \gg 1$ (long correlation scale)	$m_1 = \frac{\sigma}{4\pi}\int_{-\infty}^{\infty} d\xi\,\xi^2 S(\xi)$	$\begin{cases}\mu_1 = \operatorname{Re} m_1 = \frac{\sigma}{4\pi}\int_{-\infty}^{\infty} d\xi\,\xi^2 S(\xi) \\ \mu_2 = \operatorname{Im} m_1 = 0\end{cases}$
$\mathscr{L} \ll 1$ (short correlation scale)	$m_1 = \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} d\xi\lvert\xi\rvert S(\xi)$	$\begin{cases}\mu_1 = \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} d\xi\lvert\xi\rvert S(\xi) \\ \mu_2 = -\frac{1}{4}\sigma(\sigma-1)(3\sigma-1)S(\sigma)\end{cases}$

TABLE 1. The asymptotic expressions for m_1 in the limit of long and short correlation scales

speed and wave amplitude. To illustrate this, we shall apply our theory to two pieces of coastline with differing length scales and calculate numerically the effects of such irregularities.

The theory predicts average results for an ensemble of coastlines, under the assumption that this ensemble represents a stationary random function. To apply the theory we must therefore assume (*a*) that there exists a stationary random function $s(y)$ of which the actual coast under consideration is a sufficiently typical realization and that the theoretical average effects on Kelvin waves give a good indication of the actual effects of this coastline on Kelvin waves; (*b*) that the required statistical properties of $s(y)$ can be estimated by taking spatial averages along the actual coastline, i.e. $s(y)$ must be taken to be ergodic. (This is consistent with (*a*) since ergodicity implies stationarity.) We further assume that a spectral estimate based on a finite length of coastline is sufficiently accurate.

(i) *California coast.* The average depth of the Pacific off the California coast is taken to be 3·4 km. This gives a length scale $(gh)^{\frac{1}{2}}/f$ of 2600 km. The coast covered in our calculation is from the southern tip of Baja California to Cape Mendocino. Instead of the actual coastline, the contour of the 1000 fathoms depth line was used in the calculations as it is a better approximation of the vertical coast assumed in the theory. This contour is shown in figure 1. The direction of the axis is somewhat arbitrary, but this uncertainty should not affect the results at large ξ. The contour was digitized with 320 points and the U.B.C. BMD02T program was used to perform the spectral analysis. The results are shown in figure 2. The bars indicate the range within which 90 % of statistical variations due to the limited length of record should fall.

The distance over which the covariance function (not shown) has large values is about 0·15. Hence the equations appropriate for small $\mathscr{L}$ should be used in calculating m_1. The range of integration in (6.4) is from 0 to ∞. But as can be seen from figure 2, the contribution from very small ξ is hard to estimate. We thus calculate instead the integral

$$\frac{m_1}{\sigma} = \frac{1}{\pi}\int_{\xi}^{\infty} d\xi\,\xi S(\xi), \quad \xi > \xi_{\min} = 2\pi\hat{k}_{\min} \quad (\hat{k}_{\min} = 7{\cdot}6),$$

which gives (negative) corrections to the phase velocity due to coastal irregularities of dimension smaller than $2\pi/\xi$. Calculated values of this integral are shown

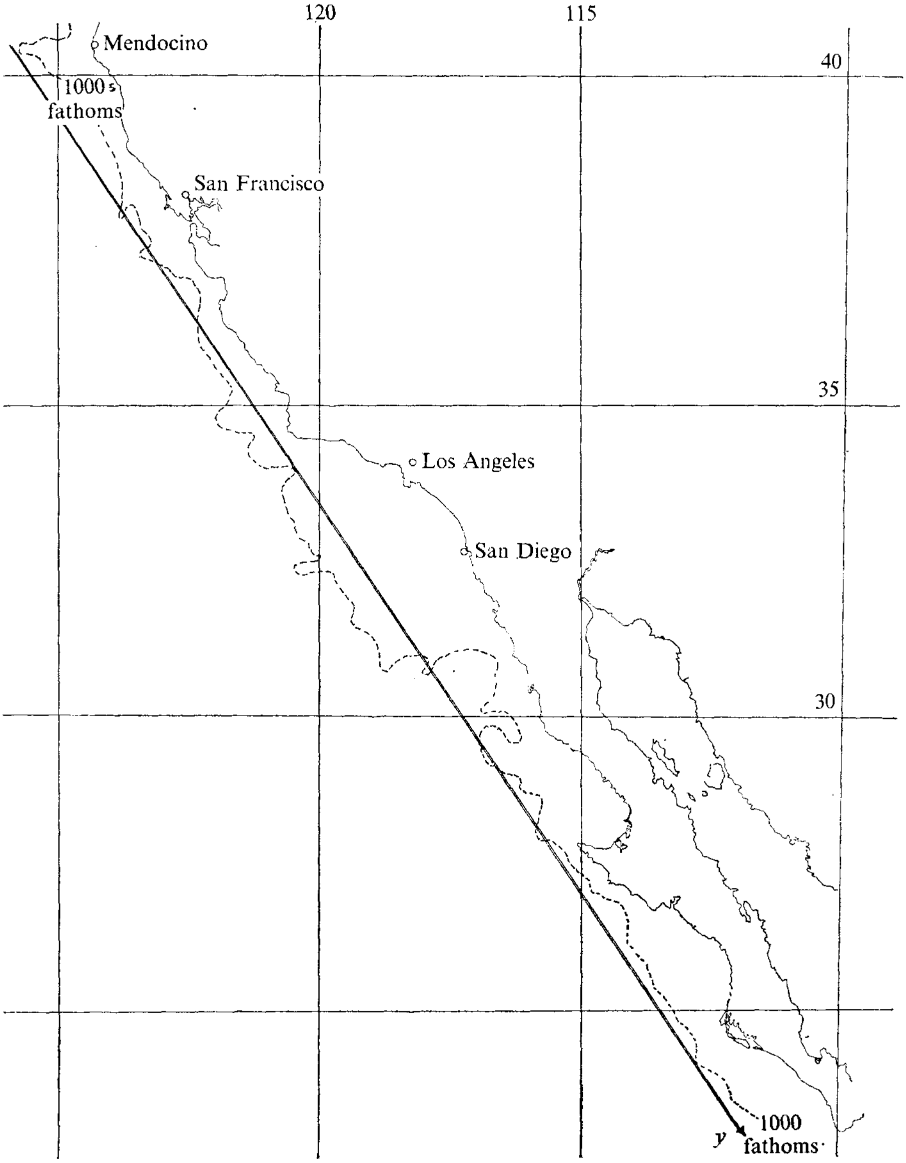

FIGURE 1. The contour of the 1000 fathom depth line from the southern tip of Baja California to Cape Mendocino.

in figure 3. By extrapolating the curves to $\hat{k} = 0$ and considering statistical error, we may conclude that the percentage decrease $|\Delta c|/c$ in phase velocity, is in the range 0·6–1·6 %. (See phase speed formulae in results 4 and 6, with $\epsilon = 1$.)

The imaginary part of m_1 is determined by the value of $S(\xi)$ at $\xi = \sigma$. For diurnal and semi-diurnal tides, $\sigma = 1$ and $\sigma = 2$ respectively. It is clear from figure 3 that the coast is too short to yield information about $S(\xi)$ at $\xi = 1$ or $\xi = 2$, corresponding to relatively long length scales. This implies that the

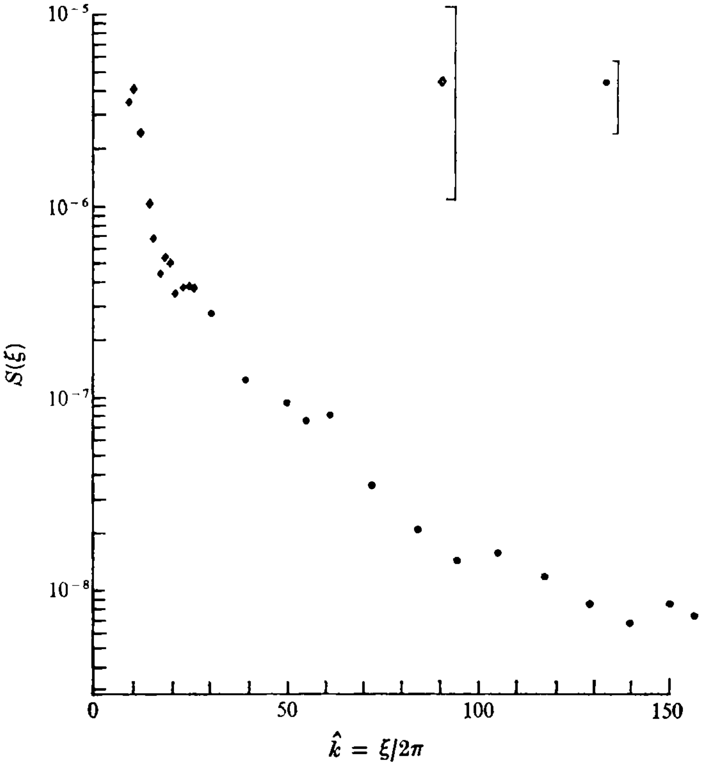

FIGURE 2. Power spectral density $S(\xi)$ of the 1000 fathom contour of the California coast. The unit of the abscissa is $\hat{k} = \xi/2\pi$. The error bars are the 90 % confidence limits.

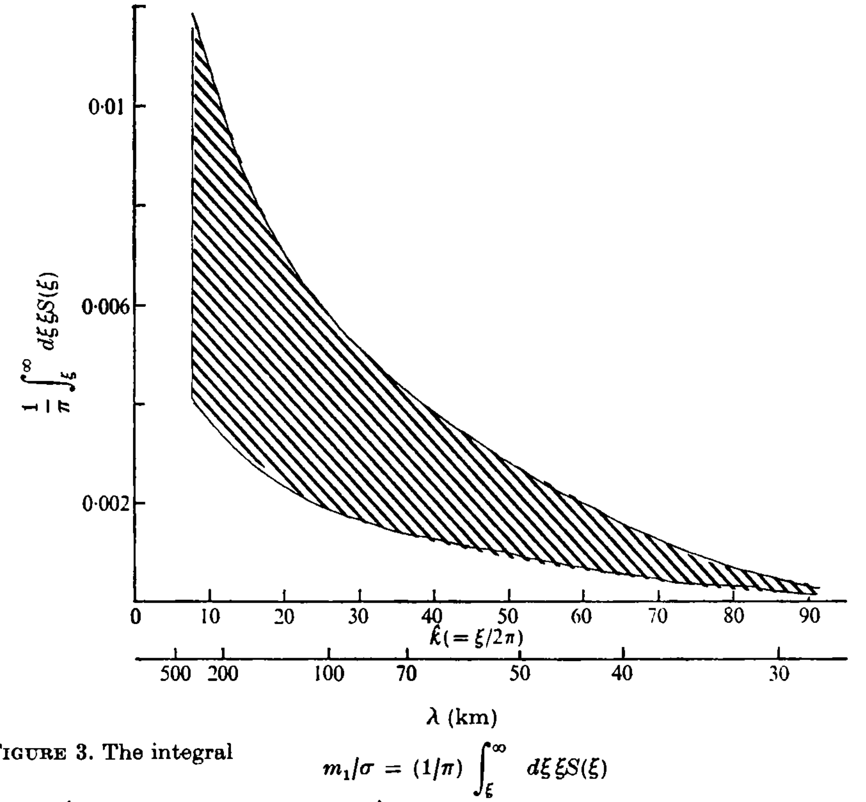

FIGURE 3. The integral

$$m_1/\sigma = (1/\pi)\int_\xi^\infty d\xi\,\xi S(\xi)$$

versus $\hat{k}$ ($= \xi/2\pi$) and $\lambda = 2600$ km$/\hat{k}$. The large range of values of this integral for each ξ or λ reflects the large statistical error in $S(\xi)$.

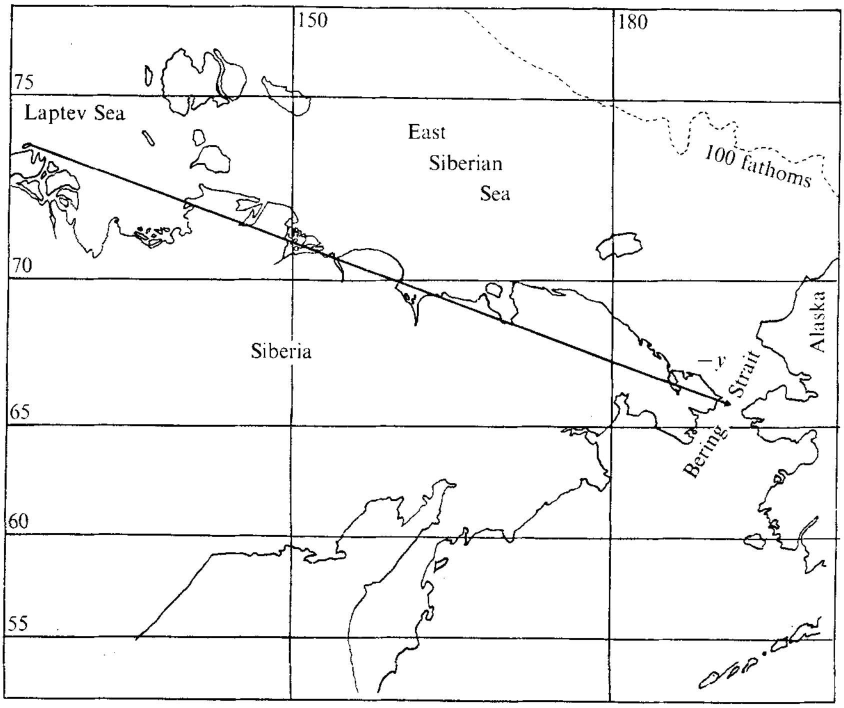

FIGURE 4. The coastline of the East Siberian Sea and Laptev Sea, covering an east–west span of about 3500 km.

energy of Kelvin waves is dissipated through scattering from large scale irregularities only. Therefore we conclude that in a relatively short length of coastline, in which large scale irregularities are absent, attenuation will likely be very small. In fact, the amplitude of a Kelvin wave is more likely to increase owing to the influx of energy from incident Poincaré waves (Howe & Mysak 1973).

(ii) *North Siberian coast.* The coastline of the East Siberia Sea and Laptev Sea from Olenekskiy Bay to the Bering Strait (see figure 4) is characterized by fairly large scale irregularities and shallow off-shore waters. The average depth of the East Siberia Sea and the Laptev Sea is about 100 m, with a corresponding length scale for Kelvin waves of 220 km.

The spectrum of the coastline is shown in figure 5. We notice that the bandwidth of wavenumbers is considerably smaller than that for the California coast. This is a result of a relatively large correlation scale of the irregularities. Because of this, the exact expressions for m_1, (4.18) and (4.19), were used to calculate the correction to the phase speed. In the case of semi-diurnal M_2 tide, $\sigma = 1{\cdot}04$, and the two branch points of the integrand in (4.18) are both near $\xi = 1$. Since the spectral analysis only covers values of ξ greater than about 2, the integration does not include the branch points.

We show in figures 6(*a*) and (*b*) values of the integral

$$\frac{m_1}{\sigma} \simeq \frac{1}{2\pi}\left(\int_{-\infty}^{-\xi} + \int_{\xi}^{\infty}\right) d\xi\, \sigma\xi S(\xi)\, A(\xi)$$

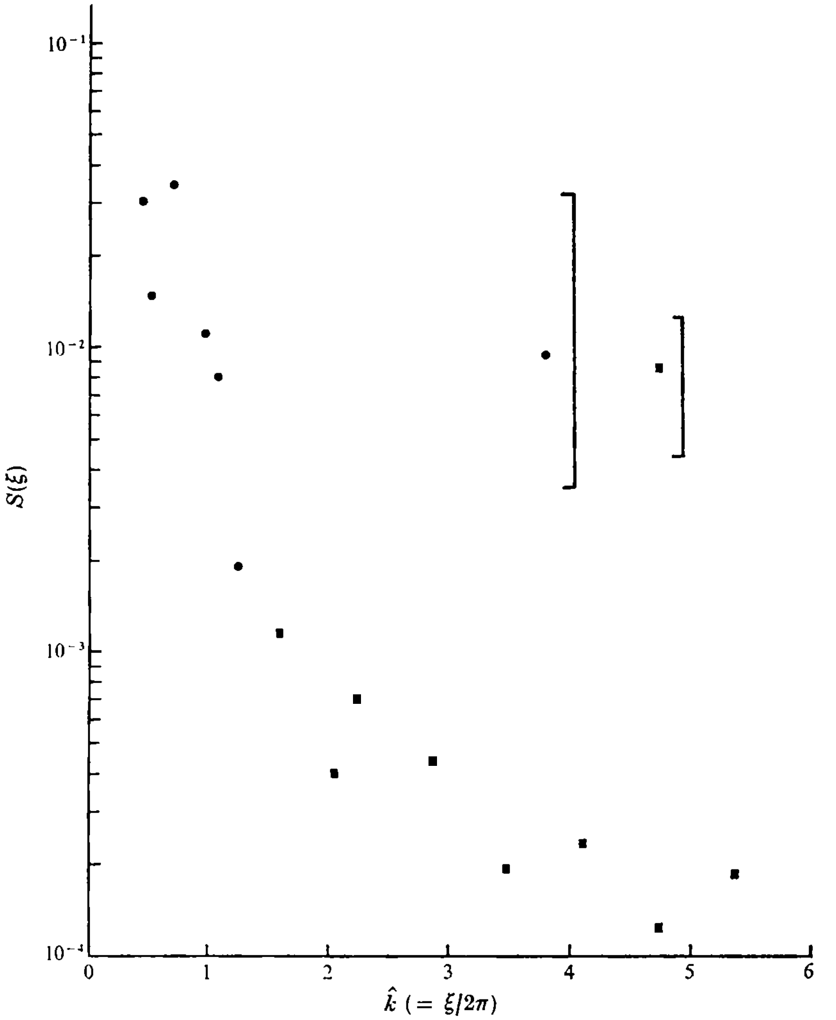

FIGURE 5. Power spectral density $S(\xi)$ of the coastline of the East Siberian Sea and Laptev Sea. The unit of the abscissa is $\hat{k} = \xi/2\pi$. The error bars are the 90 % confidence limits.

for the cases $\sigma = 0{\cdot}54$ (diurnal K_1 tide) and $\sigma = 1{\cdot}04$ (semi diurnal M_2 tide) as a function of $\hat{k}$. The figures show that the correction for the M_2 tide is greater than that for the K_1 tide. This result is expected since the M_2 tide has a shorter wavelength and hence is more sensitive to the irregularities. The upper curve for the M_2 tide is probably too large for the theory to be valid. We can, however, set a lower limit for the percentage change in phase velocity: $|\Delta c|/c \gtrsim 25\,\%$. For the K_1 tide, $|\Delta c|/c$ is in the range 18–34 %.

As in the case of the California coast, the Siberia coastline is too short to yield information about the imaginary part of m_1.

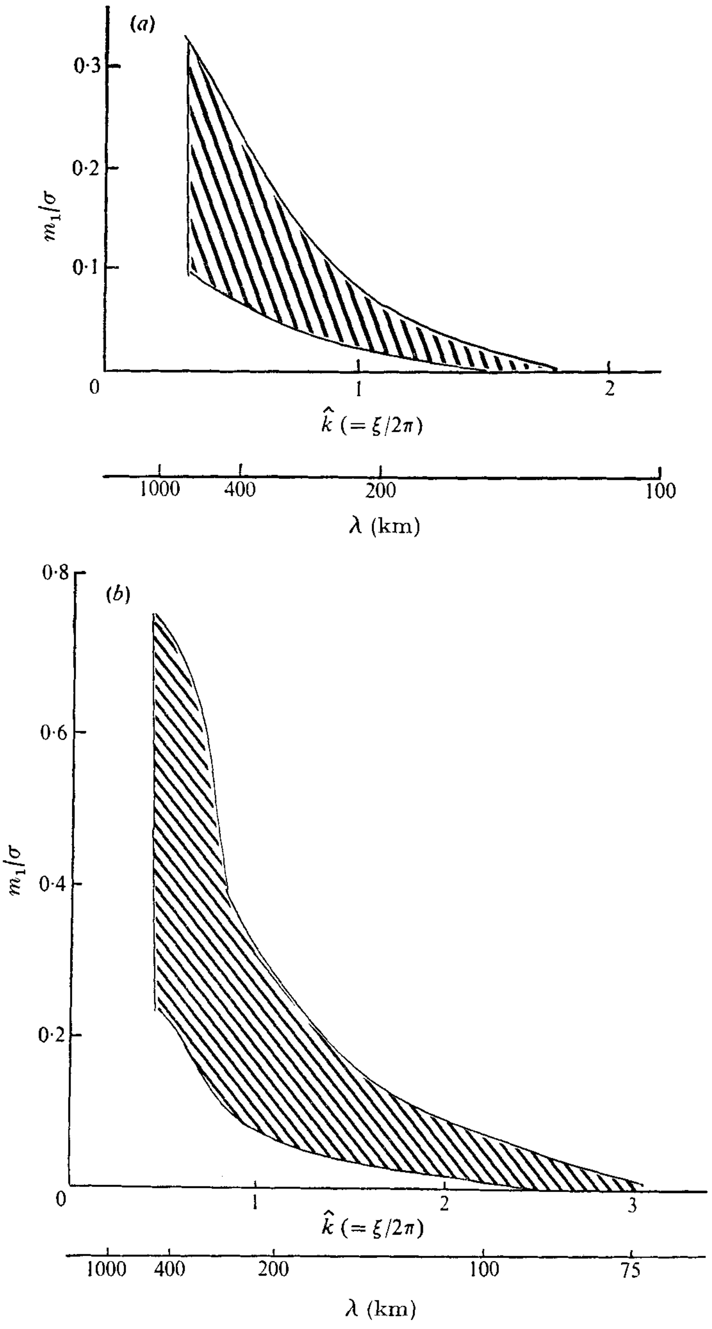

FIGURE 6. The integral

$$m_1/\sigma \simeq (1/2\pi)\left(\int_{-\infty}^{-\xi} + \int_{\xi}^{\infty}\right) d\xi\sigma\xi S(\xi)\, A(\xi)$$

versus $\hat{k}$ ($= \xi/2\pi$) and $\lambda = 220\ \text{km}/\hat{k}$ for (*a*) $\sigma = 0{\cdot}54$ and (*b*) $\sigma = 1{\cdot}04$. The large range of values of this integral for each ξ or λ reflects the large statistical error in $S(\xi)$.

8. Summary and concluding remarks

We have shown that coastal irregularities can cause a coherent Kelvin wave to slow down and to give up part of its energy in the form of random or fluctuating Poincaré and Kelvin waves. For $\omega < f$, only Kelvin waves can be generated by forward scattering from the irregularities. Although the wave amplitude remains unchanged, the group velocity and hence energy flux decrease as a result of the modification to the dispersion relation. For $\omega > f$ scattering from the irregularities can produce both Kelvin and Poincaré waves, and the energy associated with the Poincaré waves is radiated away from coast; also in this case the wave amplitude decreases with position in the direction of propagation along the coast. The size of the decrease in phase velocity and wave amplitude is determined by an integral involving the power spectrum of the coastline.

In the case of the California coast, the decrease in phase velocity is about 1 %. Measurements by Munk, Snodgrass & Wimbush (1970) yielded non-dimensional speeds of 1·1 for K_1 tides ($\sigma = 1$) and 0·7 for M_2 tides ($\sigma = 2$). Our calculations suggest that the deviations from the theoretical values for both the K_1 and M_2 tides are not due to coastal irregularities. It is well known that an isolated harbour or promontory can cause a phase shift in a Kelvin wave (Miles & Munk 1961; Miles 1972*b*), but such phase shifts cancel each other on a coast with continuous irregularities. The residual effect is of order $(\epsilon m\mathscr{L})^2$. Therefore, for any appreciable change in a Kelvin wave of a given frequency or wavenumber, either the magnitude of the irregularities or the correlation scale $\mathscr{L}$ has to be large.

The measurements also show that, while the amplitude of the K_1 tide is roughly constant, the M_2 tidal amplitude increases in the direction of propagation. Our results (see table 1) indicate that the change (i.e. decrease) in the wave amplitude is caused by large scale irregularities only. For example, for the semi-diurnal tides, the length scale which will produce a noticeable change in the amplitude is of order $2\pi/m \simeq 3$, which corresponds to a stretch from Alaska to Central America. Therefore, the presence of coastal irregularities from Baja California to Cape Mendicino do not account for the observed changes. Such changes are more likely to be due to effects such as transfer of energy from Poincaré waves (Howe & Mysak 1973), forcing by the tidal potential as modified by distortions in the sea bottom (Munk *et al.* 1970), and curvature of the earth (Miles 1972*a*).

The percentage decrease in phase speed for the Siberian coast is greater than that for the California coast by one order of magnitude. For the M_2 semi-diurnal tide, we find a lower limit of 25 % and for the K_1 diurnal tide, a decrease of about 25 %. Data for the Siberian coast are not available, but it would be interesting to see if observations do show this marked decrease from the classical value $(gh)^{\frac{1}{2}}$.

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